

THE RADON-NIKODYM THEOREM FOR THE BOCHNER INTEGRAL

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Our Main Theorem, which we believe to be the first general Radon-Nikodym theorem for the Bochner integral, is

MAIN THEOREM. *Let (X, S, μ) be a σ -finite positive measure space and let B be a Banach space. Let m be a B -valued measure on S . Then m is the indefinite integral with respect to μ of a B -valued Bochner integrable function on X if and only if*

- (1) *m is μ -continuous, that is, $m(E)=0$ whenever $\mu(E)=0$, $E \in S$,*
- (2) *the total variation, $|m|$, of m is a finite measure,*
- (3) *locally m somewhere has compact average range, that is, given $E \in S$ with $0 < \mu(E) < \infty$ there is an $F \subseteq E$ such that $\mu(F) > 0$ and*

$$A_F(m) = \{m(F')/\mu(F') : F' \subseteq F, \mu(F') > 0\}$$

is relatively (norm) compact, or equivalently

- (3') *locally m somewhere has compact direction, that is, given $E \in S$ with $0 < \mu(E) < \infty$ there is an $F \subseteq E$ and a compact subset, K , of B not containing 0 such that $\mu(F) > 0$ and $m(F')$ is contained in the cone generated by K for all $F' \subseteq F$.*

Hypothesis 3 is the traditional type of hypothesis which has been used in previous Radon-Nikodym theorems for vector-valued measures. We discuss these previous theorems below. But it is not obvious that hypothesis (3) is satisfied by every real-valued measure, and so, if hypothesis (3) is used, the classical Radon-Nikodym theorem is not an immediate consequence of the Main Theorem. Hypothesis (3'), on the other hand, is clearly satisfied by every real-valued measure, or, more generally, by every measure with values in a finite dimensional Banach space (let K be the entire surface of the unit ball of B). Also, hypothesis (3') can be interpreted geometrically as follows: the role of K is to specify a solid angle in B (in fact K can be taken to be a compact subset of the surface of the unit sphere in B , if desired), and hypothesis (3') says that locally there are sets of positive measure on subsets of which the directions of the values of m lie in this solid angle.

We feel that the methods used in proving the Main Theorem are at least as interesting as the theorem itself. Our proof is entirely elementary in the sense that we use only very basic facts about Banach spaces. In particular, we do not use

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linear functionals, much less the Hahn-Banach theorem (we do use the Kreĭn-Milman theorem for *norm* compact subsets of a Banach space, but we give an elementary proof of this theorem for this special case (§4)). The crux of our proof is an analogue for vector-valued measures (Theorems 2.4 and 2.7) of the Hahn decomposition theorem for real-valued measures. This result may be of independent interest.

Finally, we indicate how the Main Theorem can be used to give new proofs of several previously obtained Radon-Nikodym theorems, even for weakly integrable functions with values in a locally convex topological vector space. This is done by showing that if K is a subset of a locally convex space whose relative topology is compact and has a countable base, then a norm can be defined on the linear manifold spanned by K in such a way that on K the norm topology coincides with the original topology (Theorem 5.2). This result may be useful in other situations in which it is desired to apply results about Banach spaces to locally convex spaces.

We now discuss briefly the relation between the hypotheses of the Main Theorem and the hypotheses of Radon-Nikodym theorems obtained previously. Hypotheses (1) and (2) and the fact that the measure space must be assumed σ -finite [10, p. 131, Exercise 8] are familiar from the real-valued case, although for vector-valued measures hypothesis (2) is considerably stronger than just the requirement that the measure be finite. As has been mentioned, hypothesis (3) is the traditional type of hypothesis which has been used in previous Radon-Nikodym theorems for vector-valued measures. The key hypothesis of the Dunford-Pettis theorem [7, Theorem 2.1.1] is equivalent to the assumption that $A_X(m)$ is a bounded, and so relatively w^* compact, subset of the dual of a separable Banach space. In Phillips' theorem [13, p. 130] it is assumed that $A_X(m)$ is a relatively weakly compact subset of a Banach space. A similar condition is used by Rao [14]. In the theorems of Dunford-Schwartz [8, p. 499] and Bourbaki [3, p. 46, Corollary 3] the main hypothesis is essentially that $A_X(m)$ is a relatively compact subset of a locally convex space satisfying certain separability hypotheses. A similar condition has been used by Dubins [5, p. 291]. Dieudonné [4], in giving a generalization of the Dunford-Pettis theorem, seems to have been the first to have required compactness of A_E only for certain E (although Phillips had considered a somewhat analogous condition). Metivier [12, p. 334] has generalized the theorem of Bourbaki in much the same way. We have not seen any analogues of hypothesis (3').

Because the hypotheses of the theorems of Dunford-Pettis, Dieudonné, Dunford-Schwartz, and Bourbaki explicitly include separability assumptions, and because separability is a consequence (nontrivial) of the hypotheses of Phillips' theorem (see [9, p. 426] or [3, p. 95, Exercise 24]⁽²⁾), our Main Theorem can be applied, via Theorem 5.2 mentioned above, to give new proofs of these theorems. Of

⁽²⁾ *Added in proof.* See also M. Metivier, *Martingales à valeurs vectorielles; application à la dérivation*, Ann. Inst. Fourier (Grenoble) (to appear).

course conversely, Phillips' theorem could be used to prove our Main Theorem, but such a proof could hardly be called elementary in the sense which we have been using. More recently A. and C. Ionescu Tulcea [16], [17] have shown, by quite delicate arguments depending essentially on Zorn's Lemma, that the separability assumptions in the theorems mentioned above are not needed. We consider it very unlikely that our Main Theorem can be used to prove these theorems in the absence of separability assumptions. Also our results seem to have little in common with the Radon-Nikodym theorem of Rickart [15] in which the objects to be integrated are no longer single-valued functions.

Our results are organized in the following way. In §1 we obtain certain properties of measurable and integrable functions in order to show the necessity of the hypotheses of the Main Theorem and also to motivate certain points in the proof of sufficiency. §2 is devoted to the statement and proof of our analogue of the Hahn decomposition theorem. In §3 the proof of the Main Theorem is given, and hypotheses (3) and (3') are shown to be equivalent. The elementary proof of the Kreĭn-Milman theorem for norm compact sets is given in §4. We indicate in §5 how our Main Theorem can be used to prove previously obtained Radon-Nikodym theorems, as mentioned above. Finally, in §6 we collect a few examples pertinent to our results. In particular these examples show the independence of the hypotheses of the Main Theorem.

In a future paper we hope to be able to use techniques analogous to those introduced here to give a direct proof of a slight generalization of the Radon-Nikodym theorem of Phillips⁽³⁾.

1. Measurable and integrable functions. Let (X, S, μ) be a σ -finite measure space [10], and let B be a Banach space. We use the following definitions. A B -valued function, f , on X is measurable [2] if it is the pointwise limit a.e. of a sequence of B -valued simple measurable functions, and f is (Bochner) integrable if it is the limit a.e. of a mean Cauchy sequence of simple integrable functions [6]. There is no difficulty in carrying out the development of Bochner integration given in [8] using these definitions, and in the slightly more general setting of a σ -finite rather than totally σ -finite measure space.

However, since measurable functions and finite measures are carried on measurable sets, in order to prove the Main Theorem it is sufficient to do so for the case in which (X, S, μ) is totally σ -finite. Also, there is no loss of generality in assuming that (X, S, μ) is complete. Since the exposition becomes slightly simpler if we assume that (X, S, μ) is totally σ -finite and complete, we will make this assumption throughout, although we will state some of the more important results in the more general setting.

⁽³⁾ *Added in proof.* Our not entirely successful attempt appears in "Dentable subsets of Banach spaces, with application to a Radon-Nikodym theorem," in *Functional analysis*, Thompson, Washington, D.C., 1967.

Many of the results in this section are known, in one form or another, but we have not seen elementary proofs for all of them. All of our proofs are elementary, in the sense used in the introduction.

All measurable and integrable functions in this section are assumed to have values in B .

PROPOSITION 1.1. *If f is a measurable function, then f is locally almost compact valued, that is, given $E \in S$ with $\mu(E) < \infty$, and given $\varepsilon > 0$, there is an $F \in S$, $F \subseteq E$ such that $\mu(E - F) < \varepsilon$ and $f(F)$ is a (norm) relatively compact subset of B .*

Proof. Let f_n be a sequence of simple measurable functions converging to f a.e. By Egoroff's theorem [8, p. 149] f_n converges to f almost uniformly on E , that is, there is an $F \in S$, $F \subseteq E$ such that $\mu(E - F) < \varepsilon$ and f_n converges to f uniformly on F . We show that $f(F)$ is totally bounded. Given $r > 0$, choose n such that $\|f(x) - f_n(x)\| \leq r$ for $x \in F$. Let $b_1 \cdots b_k$ be the elements of the range of f_n . Then it is clear that the balls of radius r about $b_1 \cdots b_k$ cover $f(F)$.

DEFINITION 1.2. Let f be a measurable function, and let $E \in S$. Then the essential range of f restricted to E , $\text{er}_E(f)$, is defined to be the set of those $b \in B$ such that for every $\varepsilon > 0$ the measure of $\{x \in E : \|f(x) - b\| < \varepsilon\}$ is strictly positive.

We list some obvious properties of essential ranges.

PROPOSITION 1.3. *If f and g are measurable functions, and if $E, F \in S$, then*

- (1) $\text{er}_E(f)$ is a closed subset of B ,
- (2) $\text{er}_E(f)$ is contained in the closure of the range of f restricted to E .
- (3) If $f = g$ a.e. on E , then $\text{er}_E(f) = \text{er}_E(g)$.
- (4) If $\mu(E) = 0$, then $\text{er}_E(f)$ is empty.
- (5) If $F \subseteq E$, then $\text{er}_F(f) \subseteq \text{er}_E(f)$.

COROLLARY 1.4. *If f is a measurable function, then f is locally almost essentially compact valued, that is, given $E \in S$ with $\mu(E) < \infty$, and given $\varepsilon > 0$, there is an $F \in S$, $F \subseteq E$ such that $\mu(E - F) < \varepsilon$ and $\text{er}_F(f)$ is compact.*

We remark that we have not yet shown that $\text{er}_E(f)$ is not empty if $\mu(E) > 0$. Hence the importance of

LEMMA 1.5. *Let f be a measurable function and let $E \in S$. If $\mu(E) > 0$, then $\text{er}_E(f)$ meets the range of f restricted to E .*

Proof. (This proof, simpler than our original proof, was found simultaneously by a number of my students.) By removing a suitable null set we can assume that $f(E)$ is separable. Suppose that $\text{er}_E(f) \cap f(E)$ is empty. Then for each $x \in E$ there is an $\varepsilon_x > 0$ such that

$$\mu(\{y \in E : \|f(y) - f(x)\| < \varepsilon_x\}) = 0.$$

The balls of center $f(x)$ and radius ε_x cover $f(E)$, and so, since $f(E)$ is a separable

metric space, there is a countable subcollection of these balls, with centers $f(x_n)$, which still cover $f(E)$. Then

$$E \subseteq \bigcup_{n=1}^{\infty} \{y \in E : \|f(y) - f(x_n)\| < \varepsilon_{x_n}\}$$

so that $\mu(E) = 0$.

COROLLARY 1.6. *If f is a measurable function, and $E \in S$, then*

$$N = \{x \in E : f(x) \notin \text{er}_E(f)\}$$

is a null set.

Proof. If $\mu(N) > 0$ then by Lemma 1.5 there is an $x \in N$ such that $f(x) \in \text{er}_N(f) \subseteq \text{er}_E(f)$ which contradicts the definition of N .

It is not difficult to use this result to prove

COROLLARY 1.7. *If f is measurable and $E = \bigcup_{n=1}^{\infty} E_n$, $E_n \in S$, then*

$$\text{er}_E(f) = \text{closure}\left(\bigcup_{n=1}^{\infty} \text{er}_{E_n}(f)\right).$$

LEMMA 1.8. *Let f and g be measurable functions, and suppose that $\|f(x) - g(x)\| \leq d$ a.e. on $E \in S$. Let $b \in \text{er}_E(f)$. Then*

$$\text{distance}(b, \text{er}_E(g)) \leq d.$$

Proof. Given $\varepsilon > 0$, let

$$F = \{x \in E : \|b - f(x)\| < \varepsilon, \text{ and } \|f(x) - g(x)\| \leq d\}.$$

Then $\mu(F) > 0$. By Lemma 1.5 there is a $y \in F$ such that $g(y) \in \text{er}_F(g)$. But $\|b - g(y)\| < d + \varepsilon$. Since ε is arbitrary, the proof is completed.

We will use the following notation. If $D \subseteq B$ then $c(D)$ will denote the convex hull of D [8, p. 414], $\bar{c}(D)$ will denote the closed convex hull of D , and $\text{cone}(D)$ will denote the cone with vertex 0 generated by D . If f is an integrable function then the indefinite integral of f , μ_f , is the B -valued measure defined by

$$\mu_f(E) = \int_E f d\mu, \quad E \in S.$$

For the basic properties of indefinite integrals we refer the reader to [8, p. 114].

PROPOSITION 1.9 (THE MEAN-VALUE THEOREM). *If f is an integrable function, and if $E \in S$ is such that $0 < \mu(E) < \infty$, then*

$$\mu_f(E)/\mu(E) \in \bar{c}(\text{er}_E(f)),$$

so that

$$A_E(\mu_f) \subseteq \bar{c}(\text{er}_E(f)),$$

where

$$A_E(\mu_f) = \{\mu_f(F)/\mu(F) : F \in S, F \subseteq E, 0 < \mu(F)\},$$

the average range of μ_f on E .

Proof. Given $\varepsilon > 0$ we show that the distance from $\mu_f(E)/\mu(E)$ to $\bar{c}(\text{er}_E(f))$ is less than ε . Since μ_f is absolutely μ -continuous [8, p. 114], we can find $\delta > 0$ such that if $\mu(F) < \delta$ then $\|\mu_f(F)\| < \mu(E)\varepsilon/6$.

Let f_n be a sequence of simple integrable functions converging to f a.e., and so almost uniformly on E . Choose $F \in S$, $F \subseteq E$ such that f_n converges uniformly to f on F , $\mu(E-F) < \delta$, and $1/\mu(F) < 1/\mu(E) + \varepsilon/6\|f\|_1$. Then it is easily verified that

$$\|\mu_f(E)/\mu(E) - \mu_f(F)/\mu(F)\| < \varepsilon/3.$$

Thus it suffices to show that the distance from $\mu_f(F)/\mu(F)$ to $\bar{c}(\text{er}_F(f))$ is less than $2\varepsilon/3$.

Now choose m such that $\|f(x) - f_m(x)\| \leq \varepsilon/4$ for $x \in F$. Then it is easily verified that

$$\|\mu_f(F)/\mu(F) - \mu_{f_m}(F)/\mu(F)\| < \varepsilon/3.$$

Thus it suffices to show that the distance from $\mu_{f_m}(F)/\mu(F)$ to $\bar{c}(\text{er}_F(f))$ is less than $\varepsilon/3$.

Let $b_1 \cdots b_k$ be the elements of $\text{er}_F(f_m)$, so that on F

$$f_m = \sum_{i=1}^k b_i \chi_{F_i} \quad \text{a.e.,}$$

where the F_i are disjoint, $F = \bigcup_{i=1}^k F_i$, and $\mu(F_i) > 0$ for each i . By Lemma 1.8 we can find $a_i \in \text{er}_{F_i}(f)$ such that $\|b_i - a_i\| < \varepsilon/3$ for each i . Then $\sum_{i=1}^k (\mu(F_i)/\mu(F))a_i$ is in $\bar{c}(\text{er}_F(f))$ since $\sum_{i=1}^k \mu(F_i)/\mu(F) = 1$. But it is easily verified that

$$\left\| \mu_{f_m}(F)/\mu(F) - \sum_{i=1}^k (\mu(F_i)/\mu(F))a_i \right\| < \varepsilon/3.$$

PROPOSITION 1.10. *If f is an integrable function, and if $E \in S$, then*

$$\text{er}_E(f) \subseteq \bar{A}_E(\mu_f)$$

(the closure of $A_E(\mu_f)$).

Proof. Let $b \in \text{er}_E(f)$. Then, given $\varepsilon > 0$, let $F = \{x \in E : \|b - f(x)\| < \varepsilon\}$, so that $\mu(F) > 0$. Choose $F_1 \in S$, $F_1 \subseteq F$, such that $0 < \mu(F_1) < \infty$. Then $\bar{c}(\text{er}_{F_1}(f))$ is contained in the ball of radius ε about b . But by Proposition 1.9, $\mu_f(F_1)/\mu(F_1) \in \bar{c}(\text{er}_{F_1}(f))$, and so $\|b - \mu_f(F_1)/\mu(F_1)\| \leq \varepsilon$.

COROLLARY 1.11. *If f is an integrable function, and $E \in S$, then*

$$\bar{c}(A_E(\mu_f)) = \bar{c}(\text{er}_E(f)).$$

We remark that if $\text{er}_E(f)$ is compact, then it follows from the Kreĭn-Milman theorem that every extreme point of $\bar{c}(A_E(\mu_f))$ is an element of $\text{er}_E(f)$. This observation is the motivation for a key step in the proof of our decomposition theorem.

We are now in a position to prove the necessity of the hypotheses of our Main Theorem.

PROPOSITION 1.12. *If f is an integrable function, then μ_f satisfies hypotheses (1), (2), (3) and (3') of the Main Theorem. In fact, hypothesis (3) can be strengthened to (3a) locally m almost has compact average range, that is, given $E \in S$ with $\mu(E) < \infty$, and given $\varepsilon > 0$, there is an $F \subseteq E$ such that $\mu(E - F) < \varepsilon$ and $A_F(m)$ is relatively compact.*

Proof. The fact that μ_f satisfies hypotheses (1) and (2) of the Main Theorem follows from Theorem 20 on p. 114 of [8]. With respect to hypothesis (3a), if $E \in S$ with $\mu(E) < \infty$, and if $\varepsilon > 0$ are given, then by Corollary 1.4 there is an $F \in S$, $F \subseteq E$, such that $\mu(E - F) < \varepsilon$ and $\text{er}_F(f)$ is compact. Then $\bar{c}(\text{er}_F(f))$ is also compact (see [8, p. 416] for an elementary proof). But by Proposition 1.9 $A_F(\mu_f) \subseteq \bar{c}(\text{er}_F(f))$, and so $A_F(\mu_f)$ is relatively compact.

Finally, we show that μ_f satisfies hypothesis (3'). Given $E \in S$ with $0 < \mu(E) < \infty$, choose $F_0 \subseteq E$ as above so that $\text{er}_{F_0}(f)$ is compact and $\mu(F_0) > 0$. If $\text{er}_{F_0}(f) = \{0\}$ then the range of μ_f on F_0 is $\{0\}$ which is contained in the cone generated by any single point. Otherwise, choose $b \in \text{er}_{F_0}(f)$, $b \neq 0$, and let $\delta = \|b\|/2$. Let

$$F = \{x \in F_0 : \|f(x) - b\| < \delta\},$$

so that $\mu(F) > 0$, and let $K = \bar{c}(\text{er}_F(f))$, so that K is compact and convex, and $0 \notin K$. Then $\mu_f(F')/\mu(F')$ is in K for all $F' \subseteq F$, $\mu(F') > 0$, and so $\mu_f(F')$ is in $\text{cone}(K)$ for all $F' \subseteq F$.

The proof that the hypotheses of the Main Theorem are sufficient is partly motivated by Theorem 18 of [8, p. 297]. We will prove here the part of this theorem which we need, but for the more general case of B -valued functions, and also because the proof in [8] is incorrect (the second inequality in line 4, p. 298, is false).

Following [8, p. 297] let Π denote the set of all collections, π , consisting of a finite number of disjoint elements of S with strictly positive finite measure. Then Π is essentially a directed set when $\pi_1 \geq \pi$ is defined to mean that every element of π is, except for a null set, the union of elements of π_1 .

For each $\pi \in \Pi$, and each measurable function, f , which is integrable on sets of finite measure, define a function, f_π , by

$$f_\pi = \sum_{E \in \pi} (\mu_f(E)/\mu(E))\chi_E$$

where $\mu_f(E) = \int_E f d\mu$. Each f_π is a simple integrable function, and thus is in $L^p(X, S, \mu, B)$ for $1 \leq p < \infty$.

PROPOSITION 1.13. *Let $1 \leq p < \infty$. Then for each $f \in L^p(X, S, \mu, B)$ the net $\{f_\pi\}_{\pi \in \Pi}$ converges to f in p -norm.*

Proof. Let U_π be the linear operator carrying L^p into itself defined by $U_\pi(f) = f_\pi$. We first show that $\|U_\pi\|_p \leq 1$. If $f \in L^p$, then

$$\|U_\pi(f)\|_p^p = \sum_{E \in \pi} \left\| \int_E f d\mu \right\|^p (\mu(E))^{1-p}$$

which by Hölder's inequality on each $E \in \pi$ is

$$\begin{aligned} &\leq \sum_{E \in \pi} \left[\left(\int_E d\mu \right)^{1/q} \left(\int_E \|f(x)\|^p d\mu(x) \right)^{1/p} \right]^p (\mu(E))^{1-p} \\ &= \sum_{E \in \pi} \int_E \|f(x)\|^p d\mu(x) \leq \|f\|_p^p, \end{aligned}$$

where $1/p + 1/q = 1$. Thus $\|U_\pi\|_p \leq 1$.

Now if f is a simple function in L^p , so that $f = \sum_{i=1}^k b_i \chi_{E_i}$ a.e., where the E_i are disjoint sets of strictly positive finite measure, and if we let $\pi = \{E_1 \cdots E_k\}$, then it is easily checked that $U_{\pi_1}(f) = f$ a.e. for $\pi_1 \geq \pi$. Finally, if f is any element of L^p , and if $\varepsilon > 0$ is given, let g be a simple integrable function such that $\|f - g\|_p < \varepsilon/2$. Let $g = \sum_{i=1}^k b_i \chi_{E_i}$ a.e. where the E_i are disjoint sets of strictly positive finite measure, and let $\pi = \{E_1 \cdots E_k\}$. Then

$$\|f - f_{\pi_1}\|_p \leq \|f - g\|_p + \|U_{\pi_1}(g - f)\|_p < \varepsilon$$

for $\pi_1 > \pi$.

Motivated by this result, the proof of the sufficiency of the conditions of our Main Theorem consists of showing that if for each $\pi \in \Pi$ we define a simple integrable function f_π by

$$f_\pi = \sum_{E \in \pi} (m(E)/\mu(E)) \chi_E$$

then the net $\{f_\pi\}$ is a mean Cauchy net. It will then follow easily that if f is a limit of $\{f_\pi\}$, then $m = \mu_f$.

To show that $\{f_\pi\}$ is a Cauchy net we must show that given $\varepsilon > 0$ there is a $\pi \in \Pi$ such that if $\pi_1 > \pi$ then $\|f_{\pi_1} - f_\pi\|_1 < \varepsilon$. If this statement is analysed, it can be seen that, roughly speaking, each $E \in \pi$ must have the property that the vectors $m(F)/\mu(F)$ all stay close together as F varies inside E . Our Decomposition Theorem states that if m satisfies suitable conditions, involving hypothesis (3), then a $\pi \in \Pi$ whose elements have this property can be found.

2. The Decomposition Theorem. The key concept used in the Hahn decomposition theorem is the concept of sets which are (purely) positive or (purely) negative [10, p. 120] with respect to a given measure. These are defined in terms of the order relation on the real numbers. We will replace this concept of pure set by one defined in terms of convex cones in Banach spaces. Also, we will need to consider sets which are pure for a given B -valued measure relative to a given positive

measure, this concept being defined in terms of general convex sets in Banach spaces.

DEFINITION 2.1. Let (X, S, μ) be a σ -finite measure space, and let m be a B -valued measure on S . If K is a closed convex cone in B (with vertex 0) then a measurable set E is called K -pure for m if $m(F) \in K$ for all $F \subseteq E$. If m is μ -continuous and if K is any closed convex subset of B , then a measurable set E will be called K -pure for m relative to μ , or (K, μ) -pure, if $m(F)/\mu(F) \in K$ for every $F \subseteq E$ with $0 < \mu(F) < \infty$, that is, if $A_E(m) \subseteq K$.

It is clear that any subset of a K -pure (resp. (K, μ) -pure) set is K -pure (resp. (K, μ) -pure). The most important property of pure sets, which depends on the fact that K is closed and convex, is

PROPOSITION 2.2. Any countable union of K -pure (resp. (K, μ) -pure) sets is a K -pure (resp. (K, μ) -pure) set.

Proof. Let $E = \bigcup_{i=1}^{\infty} E_i$ where each E_i is K -pure (resp. (K, μ) -pure). Since the difference of K -pure (resp. (K, μ) -pure) sets is clearly K -pure (resp. (K, μ) -pure), it suffices to consider the case in which the E_i are disjoint. Let $F \subseteq E$, with $0 < \mu(F) < \infty$ in the (K, μ) -pure case. Then $F = \bigcup_{i=1}^{\infty} F \cap E_i$, this union being disjoint.

Then in the K -pure case $m(F) = \sum_{i=1}^{\infty} m(F \cap E_i)$ where $m(F \cap E_i) \in K$ for each i . Since K is a closed convex cone, it follows that $m(F) \in K$.

In the (K, μ) -pure case we have

$$m(F)/\mu(F) = \sum_{i=1}^{\infty} (m(F \cap E_i)/\mu(F \cap E_i))(\mu(F \cap E_i)/\mu(F)),$$

(where 0 is understood for terms in which $\mu(F \cap E_i) = 0$) and so $m(F)/\mu(F)$ is expressed as an infinite convex combination of elements of K , since

$$\sum_{i=1}^{\infty} \mu(F \cap E_i)/\mu(F) = 1.$$

Since K is convex and closed, it follows that $m(F)/\mu(F)$ is itself in K .

The next result provides some additional motivation for the proofs of the Decomposition Theorem and the Main Theorem.

PROPOSITION 2.3. Let f be an integrable function, and let K be a closed convex subset of B . Then $E \in S$ is (K, μ) -pure for μ_f if and only if $\text{er}_E(f) \subseteq K$.

Proof. If $\text{er}_E(f) \subseteq K$ and if $F \subseteq E$ with $0 < \mu(F) < \infty$, then

$$\mu_f(F)/\mu(F) \in \tilde{c}(\text{er}_F(f)) \subseteq K.$$

Conversely, suppose that $b \in \text{er}_E(f)$ but $b \notin K$. Let $\varepsilon > 0$ be smaller than the distance from b to K , and let

$$F = \{x \in E : \|f(x) - b\| < \varepsilon\}.$$

Then $\mu(F) > 0$, and if $F_1 \subseteq F$ with $0 < \mu(F_1) < \infty$, then $\mu_r(F_1)/\mu(F_1)$ is in the ball of radius ε about b , and so is not in K .

DECOMPOSITION THEOREM 2.4. *Let (X, S, μ) be a σ -finite measure space, and let m be a B -valued measure on S which is μ -continuous. Let $E \in S$ and suppose that $\bar{A}_E(m)$ is compact. Let $\{U_1 \cdots U_n\}$ be a collection of open convex subsets of B which covers $\bar{A}_E(m)$. Then there is a collection, $\{E_1 \cdots E_n\}$, of measurable sets whose union is E such that E_i is $(\bar{U}_i, \mu)$ -pure for $1 \leq i \leq n$.*

It is not evident that there are any nonnull $(\bar{U}_i, \mu)$ -pure sets at all, and the crux of the proof of the theorem is the following lemma which asserts that there is at least one. The proof of the theorem will then follow by a simple exhaustion argument.

If K is the closed ball of radius δ about the point $b \in B$, it will be convenient to speak of (b, δ) -pure sets instead of K -pure sets relative to μ . We recall that if K is a subset of B , then a point of K is called an extreme point of K if it lies in the interior of no line segment contained in K .

KEY LEMMA 2.5. *Let m and E satisfy the hypotheses of Theorem 2.4, and let $\delta > 0$ be given. Then there is a $b \in \bar{A}_E(m)$ and an $F \in S$, $F \subseteq E$, such that $\mu(F) > 0$ and F is (b, δ) -pure. In fact b can be chosen to be any extreme point of $\bar{c}(A_E)$.*

Proof. The proof depends on the Kreĭn-Milman theorem for the case of norm compact subsets of a Banach space. An elementary proof of this theorem for this case is given in §4. Here we will just recall the statement of the theorem.

THEOREM 2.6 (KREĬN-MILMAN THEOREM FOR NORM COMPACT SETS). *If K is a norm compact subset of B , and if K^e is the collection of extreme points of K , then $\bar{c}(K^e) = \bar{c}(K)$. Furthermore, every extreme point of $\bar{c}(K)$ lies in K^e .*

We continue with the proof of the Key Lemma. Since $\bar{A}_E$ is compact, so is $\bar{c}(A_E)$ [8, p. 416]. Let b be any extreme point of $\bar{c}(A_E)$. From the comment following Corollary 1.11 it follows that if the Main Theorem could be used, that is, if it were known that m is the indefinite integral of an integrable function f , then $b \in \text{er}_E(f)$, and so $\{x \in E : \|f(x) - b\| < \delta\}$ would be a nonnull (b, δ) -pure set for m relative to μ . Thus under the hypotheses of the present lemma we might expect to be able to find a nonnull (b, δ) -pure set for this special choice of b .

Let $B(b, \delta)$ denote the ball of radius δ about b , and let $Q = \bar{c}(A_E - B(b, \delta))$. By Theorem 2.6 $b \notin Q$ but $b \in \bar{A}_E$. Then, since Q is closed, we can find $F_0 \subseteq E$ with $0 < \mu(F_0) < \infty$ such that $m(F_0)/\mu(F_0) \in B(b, \delta) - Q$.

Suppose that F_0 is not (b, δ) -pure. We sweep out the part of F_0 which is not (b, δ) -pure as follows. Let k_1 be the smallest integer ≥ 2 for which there exists $E_1 \subseteq F_0$ such that $\mu(E_1) \geq 1/k_1$ and $m(E_1)/\mu(E_1)$ is in Q . Let $F_1 = F_0 - E_1$ and suppose that F_1 is not (b, δ) -pure. Let k_2 be the smallest integer ≥ 2 for which there exists $E_2 \subseteq F_1$ such that $\mu(E_2) \geq 1/k_2$ and $m(E_2)/\mu(E_2)$ is in Q . Let $F_2 = F_1 - E_2$. Continuing

in this way we obtain a sequence of disjoint subsets, E_n , of F_0 and a nondecreasing sequence of integers, k_n , with the property that for each n , $\mu(E_n) \geq 1/k_n$, $m(E_n)/\mu(E_n)$ is in Q , and if $E' \subseteq F_0 - \bigcup_{i=1}^n E_i$ and $m(E')/\mu(E')$ is in Q then $\mu(E') < 1/(k_n - 1)$. Since F_0 has finite measure, k_n must converge to ∞ .

Let $E_0 = \bigcup_{n=1}^{\infty} E_n$. Then, since the E_n are disjoint,

$$m(E_0)/\mu(E_0) = \sum_{n=1}^{\infty} (m(E_n)/\mu(E_n))(\mu(E_n)/\mu(E_0)).$$

Since $\sum_{n=1}^{\infty} \mu(E_n)/\mu(E_0) = 1$, and $m(E_n)/\mu(E_n)$ is in Q for each n , and since Q is convex and closed, it follows that $m(E_0)/\mu(E_0)$ is in Q .

Let $F = F_0 - E_0$. Then F is (b, δ) -pure, for if $F' \subseteq F$, $\mu(F') > 0$, and $m(F')/\mu(F')$ is in Q , then $F' \subseteq F_0 - \bigcup_{i=1}^n E_i$ for each n , and so $\mu(F') < 1/(k_n - 1)$ for each n . Since k_n converges to ∞ , $\mu(F') = 0$, so that no such F' exists.

Finally, we note that $\mu(F) > 0$, for if $\mu(F) = 0$, then $m(F) = 0$, and so $m(F_0)/\mu(F_0) = m(E_0)/\mu(E_0)$, which is in Q , and this contradicts the way in which F_0 was chosen.

Proof of Theorem 2.4. First suppose that $\mu(E) < \infty$. For each i , $1 \leq i \leq n$, let

$$a_i = \sup \{ \mu(F) : F \subseteq E \text{ and } F \text{ is } (\bar{U}_i, \mu)\text{-pure} \}.$$

Then a_i is finite since $\mu(E) < \infty$. Let E_{ij} be a sequence of $(\bar{U}_i, \mu)$ -pure subsets of E such that $\lim_j \mu(E_{ij}) = a_i$, and let $E_i = \bigcup_{j=1}^{\infty} E_{ij}$. Then E_i is a $(\bar{U}_i, \mu)$ -pure set by Proposition 2.2, and, since it is clear that $\mu(E_i) = a_i$, it follows that E_i is a $(\bar{U}_i, \mu)$ -pure subset of maximal measure in E .

Let $F = E - \bigcup_{i=1}^n E_i$. We show that $\mu(F) = 0$. Suppose that $\mu(F) > 0$, and let b be any extreme point of $\bar{c}(A_F)$. Since b is in $\bar{A}_F$, b is contained in U_{i_0} for some i_0 , so that we can choose $\delta > 0$ such that $B(b, \delta)$ is contained in U_{i_0} . Then by the Key Lemma 2.5 there is an $F' \subseteq F$ such that $\mu(F') > 0$ and F' is (b, δ) -pure and so $(\bar{U}_{i_0}, \mu)$ -pure. Since F' is disjoint from E_{i_0} this contradicts the maximality of E_{i_0} . Thus $\mu(F) = 0$. Then we can adjoin F to any E_i , so that $E = \bigcup_{i=1}^n E_i$, where E_i is $(\bar{U}_i, \mu)$ -pure for $1 \leq i \leq n$.

Now suppose that E does not have finite measure. Then since μ is σ -finite, $E = \bigcup_{j=1}^{\infty} F_j$ where $\mu(F_j) < \infty$ for each j . Then by the first part of this proof we can find sets E_{ji} , $1 \leq i \leq n$, such that $E_j = \bigcup_{i=1}^n E_{ji}$ for each j , and E_{ji} is $(\bar{U}_i, \mu)$ -pure for each i . Let $E_i = \bigcup_{j=1}^{\infty} E_{ji}$. Then E_i is $(\bar{U}_i, \mu)$ -pure for each i by Proposition 2.2, and it is clear that $E = \bigcup_{i=1}^n E_i$.

While Theorem 2.4 is in the form which we need for the proof of the Main Theorem, it has the disadvantage that the usual Hahn decomposition theorem is not apparently a corollary. For this reason we now state the theorem in a form such that the Hahn decomposition theorem, at least for finite measures, is a corollary.

THEOREM 2.7. *Let m be a B -valued measure and let $E \in S$ be such that $|m|$ is totally σ -finite on E . Suppose that there is a compact subset, K , of B not containing 0 such that the range of m restricted to E is contained in $\text{cone}(K)$. Let $\{U_1 \cdots U_n\}$ be a collection of open convex subsets of B which cover K . Then there is a collection*

$\{E_1 \cdots E_n\}$ of measurable sets whose union is E such that E_i is $\text{cone}(\bar{U}_i)$ -pure for m for $1 \leq i \leq n$.

Proof. Let $\mu = |m|$. As in the proof of Theorem 2.4 it suffices to consider the case in which $\mu(E) < \infty$. Also by the argument used in the proof of Theorem 2.4 let $E_1 \cdots E_n$ be subsets of E of maximal measure such that E_i is $\text{cone}(\bar{U}_i)$ -pure for $1 \leq i \leq n$. Let $F = E - \bigcup_{i=1}^n E_i$.

We show that $\mu(F) = 0$. Suppose on the contrary that $\mu(F) > 0$. Then A_F is not empty, and must contain points other than 0, since otherwise $\mu(F) = |m|(F) = 0$. Furthermore $\|m(F')\| \leq |m|(F') = \mu(F')$ for any $F' \in S$, so A_F is contained in the unit ball of B . Let $1/k$ be the distance from K to 0. Then

$$A_F \subseteq \bar{c}(kK \cup \{0\})$$

so that $\bar{c}(A_F)$ is compact. Since A_F contains points other than 0, $\bar{c}(A_F)$ must have a nonzero extreme point, say b . Since b is in $\bar{A}_F$, b is contained in the interior of $\text{cone}(U_{i_0})$ for some i_0 , so that we can choose $\delta > 0$ such that $B(b, \delta)$ is contained in $\text{cone}(U_{i_0})$. Then by the Key Lemma 2.5 there is an $F' \subseteq F$ such that $\mu(F') > 0$ and F' is (b, δ) -pure relative to μ . It follows that F' is $\text{cone}(U_{i_0})$ -pure, contradicting the maximality of E_{i_0} .

We remark that once the Main Theorem has been proven, Theorem 2.7 can be considerably strengthened in that the U_i can be replaced by any countable collection of closed convex sets, $\{K_i\}$, whose union contains K . In fact, if m is the indefinite integral of f with respect to $\mu = |m|$ on E , then it suffices, except for a possible null set, to let $E_i = \{x \in E : f(x) \in \text{cone}(K_i)\}$.

COROLLARY 2.8 (HAHN DECOMPOSITION THEOREM FOR FINITE MEASURES). *If m is a finite real measure, then there exist disjoint sets E_1 and E_2 whose union is X such that E_1 is purely positive and E_2 is purely negative for m .*

Proof. Let $K = \{1, -1\}$ and $U_1 = (0, 2)$, $U_2 = (-2, 0)$. Then the only point which remains to be checked before applying Theorem 2.7 is that $|m|$ is finite. But this follows from [8, p. 97, Lemma 5]. (Of course it is important that we are defining $|m|$ by [8, p. 97, Definition 4] and not in terms of a Hahn decomposition as is done in [10].)

3. The proof of the Main Theorem. Assume first that m satisfies hypotheses (1), (2) and (3a) (see Proposition 1.12). (At the end of this section we show that hypotheses (3) and (3') are equivalent to (3a).) We now show that $\{f_n\}$, as defined at the end of §1, is a mean Cauchy net. Let $\varepsilon > 0$ be given. Since $|m|$ is a finite measure, we can find $E \in S$ such that $\mu(E) < \infty$ and $|m|(X - E) < \varepsilon/3$. Since m is μ -continuous, so is $|m|$. Then, since $|m|$ is a finite measure, it is absolutely μ -continuous, [10, p. 125] so that we can find $\delta > 0$ such that if $\mu(F) < \delta$ then $|m|(F) < \varepsilon/6$. Choose $E_0 \subseteq E$ such that $\mu(E_0) < \delta$ and $\bar{A}_{E-E_0}$ is compact.

Let $b_1 \cdots b_n$ be elements of B which are $(\varepsilon/6\mu(E))$ -dense in $\bar{A}_{E-E_0}$, that is, if U_i

denotes the open ball about b_i of radius $\varepsilon/6\mu(E)$, then the U_i cover $\bar{A}_{E-E_0}$. Then, applying the Decomposition Theorem 2.4, we can find disjoint sets $E_1 \cdots E_n$ whose union is $E - E_0$ such that E_i is $(b_i, \varepsilon/6\mu(E))$ -pure for each i . By eliminating those E_i which are null sets, by eliminating the corresponding b_i , and by adjusting the remaining E_i by null sets, we can assume that $\mu(E_i) > 0$ for each i (but the b_i need no longer be $\varepsilon/6\mu(E)$ -dense). Let $\pi_0 = \{E_i : 0 \leq i \leq n\}$, or, if $\mu(E_0) = 0$, let $\pi_0 = \{E_i : 1 \leq i \leq n\}$, so that $\pi_0 \in \Pi$.

We now show that if $\pi > \pi_0$, then $\|f_\pi - f_{\pi_0}\|_1 < \varepsilon$. We will assume that $\mu(E_0) > 0$, for it will be clear how the proof simplifies if $\mu(E_0) = 0$. If $\pi > \pi_0$, then, except for possible null sets,

$$\pi = \{F_1 \cdots F_k\} \cup \{F_{ij} : 0 \leq i \leq n, 1 \leq j \leq k_i\}$$

where the F_i do not meet E for $1 \leq i \leq k$, and $E_i = \bigcup_{j=1}^{k_i} F_{ij}$ for $0 \leq i \leq n$. Of course all the elements of π are disjoint and have strictly positive measure. Then

$$\begin{aligned} \|f_\pi - f_{\pi_0}\|_1 &= \int \|f_\pi(x) - f_{\pi_0}(x)\| d\mu(x) \\ &= \sum_{i=1}^k \|m(F_i)\| + \sum_{j=1}^{k_0} \|m(F_{0j})/\mu(F_{0j}) - m(E_0)/\mu(E_0)\| \mu(F_{0j}) \\ &\quad + \sum_{i=1}^n \left\{ \sum_{j=1}^{k_i} \|m(F_{ij})/\mu(F_{ij}) - m(E_i)/\mu(E_i)\| \mu(F_{ij}) \right\} \\ &\leq |m|(X - E) + \sum_{j=0}^{k_0} \|m(F_{0j})\| + \|m(E_0)\| \\ &\quad + \sum_{i=1}^n \left\{ \sum_{j=1}^{k_i} (\|m(F_{ij})/\mu(F_{ij}) - b_i\| + \|b_i - m(E_i)/\mu(E_i)\|) \mu(F_{ij}) \right\} \\ &\leq \varepsilon/3 + \varepsilon/6 + \varepsilon/6 + (\varepsilon/3\mu(E)) \mu\left(\bigcup_{i=1}^n E_i\right) \leq \varepsilon. \end{aligned}$$

Thus the net $\{f_\pi\}$ is a mean Cauchy net, and so converges in mean to some element, f , of $L^1(X, S, \mu, B)$. In particular,

$$\mu_f(E) = \int_E f d\mu = \lim_\pi \int_E f_\pi d\mu$$

for every $E \in S$.

We now show that $m(E) = \mu_f(E)$ for all $E \in S$. Let $E \in S$. If $\mu(E) = 0$ the result follows from the μ -continuity of m . If $0 < \mu(E) < \infty$, let $\pi_0 = \{E\}$. Then it is easily checked that

$$\int_E f_\pi d\mu = m(E)$$

whenever $\pi \geq \pi_0$, and so

$$\mu_f(E) = \lim_\pi \int_E f_\pi d\mu = m(E).$$

The result when $\mu(E) = \infty$ then follows from the σ -finiteness of μ .

We now show that hypotheses (3) and (3') are equivalent to hypothesis (3a) in the presence of hypotheses (1) and (2). Clearly hypothesis (3a) implies hypothesis (3). Also, if m satisfies hypothesis (3a), then, by the part of the Main Theorem proved so far, $m = \mu_f$ for some Bochner integrable f , and so m satisfies hypothesis (3') by Proposition 1.12 (a direct proof can also be given).

Now let m satisfy hypothesis (3'). Then we show that m satisfies hypothesis (3). Given $E \in S$ with $0 < \mu(E) < \infty$, choose $F_0 \subseteq E$ and compact $K \subseteq B$ not containing 0 such that $\mu(F_0) > 0$ and the range of m on F_0 is contained in $\text{cone}(K)$. Choose a constant c large enough so that the measure $c\mu - |m|$ is not purely negative on F_0 , and let F be the purely positive part of a Hahn decomposition (Corollary 2.8) of $c\mu - |m|$. Then $\mu(F) > 0$, and $\|m(F')/\mu(F')\| < c$ for $F' \subseteq F$ and $\mu(F') > 0$. It follows that if $1/k$ is the distance from K to 0, then $m(F')/\mu(F')$ is in $\bar{c}(ckK \cup \{0\})$ which is compact, and so hypothesis (3) is satisfied.

It remains to show that if m satisfies hypothesis (3), then m satisfies hypothesis (3a). First we make the easily checked observation that $A_{E \cup F} \subseteq c(A_E \cup A_F)$ for any $E, F \in S$, so that $\bar{A}_{E \cup F}$ is compact if $\bar{A}_E$ and $\bar{A}_F$ are. Now let m satisfy hypothesis (3), and let $E \in S$ with $\mu(E) < \infty$ be given. Let

$$a = \sup \{ \mu(F) : F \subseteq E \text{ and } \bar{A}_F \text{ is compact} \}.$$

It suffices to show that $a = \mu(E)$. Let $\{F_i\}$ be a sequence of subsets of E such that $\mu(F_i) \rightarrow a$ and $\bar{A}_{F_i}$ is compact. By the observation made above we can assume that the F_i are increasing. Let $E_0 = \bigcup F_i$, so that $\mu(E_0) = a$. If $a < \mu(E)$, then $\mu(E - E_0) > 0$, so that by hypothesis (3) there is an $F_0 \subseteq E - E_0$ such that $\mu(F_0) > 0$ and $\bar{A}_{F_0}$ is compact. Then $\bar{A}_{F_i \cup F_0}$ is compact, and $\mu(F_i \cup F_0) \rightarrow a + \mu(F_0)$, contradicting the definition of a .

4. The Kreĭn-Milman theorem. We now supply our elementary proof of Theorem 2.6. The crux of the matter is to show that there exists even one extreme point.

LEMMA 4.1. *Let K be a norm compact subset of B . Then K has at least one extreme point.*

Proof. Since K is a compact metric space, we can find a sequence, $\{b_n\}$, which is dense in K . We define subsets, K_n , of K by induction. First, let $K_0 = K$. Then, if K_{n-1} has been defined, we define K_n by

$$K_n = \{x \in K_{n-1} : \|x - b_n\| = \sup \{\|y - b_n\| : y \in K_{n-1}\}\}.$$

A simple compactness argument shows that K_n is nonempty. Also, K_n is clearly closed and so compact, and $K_n \subseteq K_{n-1}$. Let $K' = \bigcap_{n=1}^{\infty} K_n$. Then K' is not empty, since K is compact. We show that K' contains exactly one point, and that this point is an extreme point of K .

Suppose that K' contained two distinct points, c and c' . Since $\{b_n\}$ is dense in K , there is an n such that

$$\|c' - b_n\| < \|c - b_n\|.$$

Then, since $c \in K_{n-1}$,

$$\|c' - b_n\| < \sup \{\|y - b_n\| : y \in K_{n-1}\},$$

and so $c' \notin K_n$, which contradicts the assumption that $c' \in K'$.

Let e be the unique point in K' , and suppose that e is not an extreme point of K , so that

$$e = tc_1 + (1-t)c_2, \quad 0 < t < 1,$$

for some $c_1, c_2 \in K$. Let n be the smallest integer such that either c_1 or c_2 is not in K_n , so that c_1 and c_2 are both in K_{n-1} . Then

$$\|c_i - b_n\| \leq \sup \{\|y - b_n\| : y \in K_{n-1}\}, \quad i = 1, 2,$$

with strict inequality holding for $i=1$ or 2 . Then

$$\begin{aligned} \|e - b_n\| &\leq t\|c_1 - b_n\| + (1-t)\|c_2 - b_n\| \\ &< \sup \{\|y - b_n\| : y \in K_{n-1}\}, \end{aligned}$$

which contradicts the fact that $e \in K_n$. Thus e must be an extreme point of K .

Proof of Theorem 2.6. If K^e is the set of all extreme points of K , it is clear that $\bar{c}(K^e) \subseteq \bar{c}(K)$. Suppose that $\bar{c}(K^e) \neq \bar{c}(K)$, so that $K \not\subseteq \bar{c}(K^e)$. Let

$$d = \sup \{\text{distance}(y, \bar{c}(K^e)) : y \in K\},$$

so that $d > 0$, and let

$$K' = \{x \in K : \text{distance}(x, \bar{c}(K^e)) = d\}.$$

Then K' is nonempty, compact, and disjoint from $\bar{c}(K^e)$.

Then K' has an extreme point, e , by Lemma 4.1. We show that e is also an extreme point of K , contradicting the fact that K^e contains all the extreme points of K . Suppose that e is not an extreme point of K , so that

$$e = tb_1 + (1-t)b_2, \quad 0 < t < 1,$$

for some $b_1, b_2 \in K$. Since e is an extreme point of K' , at least one of b_1, b_2 is not in K' , say b_1 , so that $\text{distance}(b_1, \bar{c}(K^e)) < d$. By the compactness of $\bar{c}(K^e)$ there exist $c_1, c_2 \in \bar{c}(K^e)$ such that

$$\|b_i - c_i\| = \text{distance}(b_i, \bar{c}(K^e)), \quad i = 1, 2.$$

In particular $\|b_1 - c_1\| < d$ and $\|b_2 - c_2\| \leq d$. Let $c = tc_1 + (1-t)c_2$. Then $c \in \bar{c}(K^e)$, and

$$\|e - c\| \leq t\|b_1 - c_1\| + (1-t)\|b_2 - c_2\| < d,$$

contradicting the fact that $e \in K'$.

The final part of Theorem 2.6 has a standard elementary proof. See for example [8, p. 440, Lemma 5].

5. Other Radon-Nikodym theorems. In this section we indicate how our Main Theorem can be used to give new proofs of the Radon-Nikodym theorems of Dunford and Pettis [7], Phillips [13, p. 134], Dieudonné [4], Bourbaki [3, p. 46], and (with separability hypotheses) Metivier [12, p. 334]. We will not actually supply proofs of all these theorems here, but instead we will give a proof of a version of Theorem 2 of [8, p. 499]. Using this proof as a model, the reader may supply his own proofs of the above mentioned theorems.

We recall that if V is a locally convex topological vector space, with dual space V^* , and if (X, S, μ) is a positive σ -finite measure space, then a function, f , from X to V is said to be weakly measurable if $\langle f(\cdot), z \rangle$ is measurable for all $z \in V^*$, and f is said to be weakly integrable if $\langle f(\cdot), z \rangle$ is integrable for all $z \in V^*$ (where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between V and V^*).

We now state Theorem 2 of [8, p. 499] explicitly in the form of a Radon-Nikodym theorem.

THEOREM 5.1. *Let (X, S, μ) be a positive finite measure space, and let V be a locally convex topological vector space. Let m be a V -valued measure on S which is μ -continuous, and suppose that $\bar{A}_X(m)$ is a compact subset of V whose relative topology has a countable base. Then there is a weakly integrable function, f , whose range is contained in $\bar{A}_X$, such that*

$$\langle m(E), z \rangle = \int_E \langle f(x), z \rangle d\mu(x)$$

for all $z \in V^$ and $E \in S$. That is, m is the weak indefinite integral of f .*

We have stated (and will prove) this theorem only for locally convex spaces. The reader will have no difficulty in supplying the few extra details needed to extend the theorem to the more general situation considered in [8].

Our proof of Theorem 5.1 (and the other Radon-Nikodym theorems mentioned above) depends on the following theorem, which enables us to apply to locally convex spaces the results we have obtained about Banach spaces.

THEOREM 5.2. *Let V be a locally convex topological vector space, and let K be a compact subset of V . Let K' be the closed circled convex extension of K [11, p. 14], and let W be the linear manifold spanned by K' . Then a norm can be defined on W such that on K' the norm topology coincides with the original topology if and only if the original topology on K has a countable base.*

Proof. Since the topology of a compact metric space has a countable base, it is clear that if the original topology on K coincides with a norm topology, then it has a countable base.

To prove the converse we first note that it suffices to do so in the case in which

V is complete [11, p. 62]. For let V^c be the completion of V . Then the relative topology of K in V^c will still be compact and have a countable base. Let K'' be the closed circled convex extension of K in V^c , and let W'' be the linear manifold in V^c spanned by K'' . Then $K' \subseteq K''$ and $W \subseteq W''$, and the topology of K' is just the relative topology from K'' . If a norm can be defined on W'' such that on K'' the norm topology coincides with the original topology, then the restriction of this norm to W will have the desired property that on K' the norm topology coincides with the original topology.

Now assume that V is complete, so that K' is compact [11, p. 113]. Let $C(K)$ be the Banach space of all continuous scalar-valued functions on K . By Lemma 4 of [8, p. 501] $C(K)$ is separable, since the topology of K has a countable base. Each element of V^* can be viewed as an element of $C(K)$. Since any subset of a separable metric space is separable, there is a sequence $\{z_n\}$ of elements of V^* , none of which is identically zero on K , such that for any $z \in V^*$ and any $\varepsilon > 0$ there is an n such that

$$|\langle v, z - z_n \rangle| < \varepsilon \quad \text{for all } v \in K.$$

Since $z - z_n$ is linear, this inequality persists for all $v \in K'$.

Since K' is compact, each z_n is bounded on K' . For each n let

$$|z_n| = \sup \{ |\langle v, z_n \rangle| : v \in K' \}.$$

Note that the z_n were chosen so that $|z_n| > 0$ for each n .

Define a function, $\| \cdot \|$, on W by

$$\|w\| = \sum_{n=1}^{\infty} 2^{-n} |z_n|^{-1} |\langle w, z_n \rangle|.$$

If $w \in K'$, then $|\langle w, z_n \rangle| \leq |z_n|$ for each n , so the sum converges. Since K' is circled and convex, each element of W is just a scalar multiple of an element of K' , so the sum converges for each element of W .

It is then clear that $\| \cdot \|$ is a seminorm on W . We now show that it is a norm. If $w \in K'$ and $w \neq 0$, then by the Hahn-Banach theorem there is a $z \in \bar{V}$ such that $|\langle w, z \rangle| \geq 1$. Then n can be chosen so that z_n is within $1/2$ of z on K , and so on K' , so that $|\langle w, z_n \rangle| \geq 1/2$. Thus $\|w\| > 0$. Since every element of W is a scalar multiple of an element in K' , it follows that $\|w\| > 0$ whenever $w \in W$ and $w \neq 0$.

We now show that on K' the norm topology coincides with the original topology from V . Suppose that w_a is a net of elements of K' which converges to $w \in K'$ in the original topology, and so in the weak topology. Given $\varepsilon > 0$ choose N large enough so that $\sum_{n>N} 2^{-n} < \varepsilon/4$. Choose a_0 so that if $a > a_0$ then

$$|\langle w - w_a, z_n \rangle| \leq |z_n| \varepsilon / 2$$

for $1 \leq n \leq N$. Then, if $a > a_0$,

$$\|w - w_a\| \leq \sum_{n=1}^N 2^{-n} \varepsilon / 2 + 2 \sum_{n>N} 2^{-n} \leq \varepsilon.$$

Thus on K' the norm topology is coarser than the original topology. But the original topology is compact and the norm topology is Hausdorff, so the two topologies must coincide.

COROLLARY 5.3. *If K is a compact subset of a locally convex space whose relative topology has a countable base, then the relative topology of the closed circled convex extension of K has a countable base.*

The following corollary is the particular version of Theorem 5.2 which is used to prove the Radon-Nikodym theorems of Dunford and Pettis [7] and Dieudonné [4] using our Main Theorem.

COROLLARY 5.4. *Let B be a separable Banach space with dual space B^* . Then a norm, $\|\cdot\|_0$, can be defined on B^* such that on bounded (and so relatively w^* -compact) subsets of B^* the $\|\cdot\|_0$ -topology coincides with the w^* -topology.*

Actually this corollary may be proved directly. If $\{b_n\}$ is a sequence of elements dense in the unit ball of B , then

$$\|b^*\|_0 = \sum_{n=1}^{\infty} 2^{-n} |\langle b_n, b^* \rangle| \quad \text{for } b^* \in B^*$$

is such a norm.

The following corollary is the particular version of Theorem 5.2 which is used to prove the Radon-Nikodym theorem of Phillips [13] using our Main Theorem (but one must first prove the nontrivial fact that the hypotheses of Phillips' theorem imply separability, see [9, p. 426] or [3, p. 95, Exercise 24]).

COROLLARY 5.5. *Let B be a separable Banach space and let K be a weakly compact subset of B . If W is the linear manifold spanned by the closed circled convex extension K' of K , then a norm, $\|\cdot\|_0$, can be defined on W such that on K' the $\|\cdot\|_0$ -topology coincides with the weak topology.*

Proof. By Theorem 3 of [8, p. 434] the weak topology of K is metrizable, and so has a countable base.

We remark that Corollaries 5.4 and 5.5 in conjunction with our proof in §4 of the Kreĭn-Milman theorem for norm compact sets yield a new proof of the Kreĭn-Milman theorem for w^* -compact subsets in the dual of a separable Banach space, and for weakly compact subsets of a separable Banach space.

Proof of Theorem 5.1. We can assume that μ is totally finite.

Let W be the linear manifold spanned by the closed circled convex extension, K , of A_X . By Theorem 5.2 there is a norm, $\|\cdot\|$, on W such that on K the original topology coincides with the norm topology. Let W_0 be the completion of W with respect to this norm, so that W_0 is a Banach space. Since $\bar{A}_X$ is compact in W it is closed in W_0 . Then m can be viewed as a measure with values in $\mu(X)K \subseteq W_0$ clearly satisfying hypotheses (1) and (3) of the Main Theorem. Hypothesis (2) is

also satisfied, for $\bar{A}_X(m)$ is bounded and so there exists a constant, N , such that

$$\|m(E)\| \leq N\mu(E) \quad \text{for } E \in S.$$

Then $|m|(X) \leq N\mu(X)$, and since μ is totally finite, $|m|$ is also.

Then, applying the Main Theorem, there is a W_0 -valued Bochner integrable function, f , such that $m = \mu_f$. Since according to Proposition 1.10 $\text{er}_X f \subseteq \bar{A}_X(m)$, we can, by Proposition 1.6, change f on a null set so that $\text{range}(f) \subseteq \bar{A}_X \subseteq V$. Furthermore, by Proposition 1.13, if, as before, we define

$$f_\pi = \sum_{E \in \pi} (m(E)/\mu(E))\chi_E$$

for $\pi \in \Pi$, then f_π converges to f in mean. Then we can extract a subsequence, $f_{\pi_n} = f_n$ which converges to f a.e. as well as in mean. In particular $\mu_{f_n}(E)$ converges in norm to $\mu_f(E) = m(E)$ for each $E \in S$.

We now show that for each $z \in V^*$ and each $E \in S$

$$\langle m(E), z \rangle = \int_E \langle f(x), z \rangle d\mu(x).$$

If $\mu(E) = 0$ the result is clear, so we may assume that $\mu(E) > 0$. Since on $\bar{c}(A_X)$ the norm topology coincides with the original topology, z is continuous on $\bar{c}(A_X)$, and so is bounded on $\bar{c}(A_X)$, say by C . It follows that $\langle f(\cdot), z \rangle$ is an integrable function bounded by C , as is $\langle f_n(\cdot), z \rangle$ for each n , since $\text{range}(f_n) \subseteq \bar{A}_X(m)$. In particular, f is weakly integrable. Furthermore, since f_n converges to f a.e., it follows that

$$\langle f_n(x), z \rangle \rightarrow \langle f(x), z \rangle \quad \text{a.e. } x.$$

Then by the Lebesgue dominated convergence theorem

$$\int_E \langle f_n(x), z \rangle d\mu(x) \rightarrow \int_E \langle f(x), z \rangle d\mu(x)$$

for each $E \in S$. Now, since the f_n are simple functions, it is easily verified that

$$\int_E \langle f_n(x), z \rangle d\mu(x) = \langle \mu_{f_n}(E), z \rangle.$$

Then, since $\mu_{f_n}(E) \rightarrow m(E)$ in norm, since $\mu_{f_n}(E)/\mu(E) \in \bar{c}(A_X)$ and since z is continuous on $\bar{c}(A_X)$, it follows that

$$\langle \mu_{f_n}(E), z \rangle \rightarrow \langle m(E), z \rangle.$$

Thus

$$\langle m(E), z \rangle = \int_E \langle f(x), z \rangle d\mu(x)$$

for all $E \in S$ and $z \in V^*$, concluding the proof.

6. Examples. In this section we gather together a few examples which illustrate certain aspects of our results. In particular, they show that the hypotheses of the

Main Theorem are independent. The first three examples are well known. In all of the examples $X=[0, 1]$ and μ is Lebesgue measure on X .

Our first example shows that hypotheses (1) and (2) of the Main Theorem do not imply hypothesis (3) (or 3')).

EXAMPLE 6.1. Let $B=L^1(X, \mu)$, and define a B -valued measure, m , by $m(E)=\chi_E$ for each Lebesgue measurable set E . Then m is μ -continuous, in fact, $|m|=\mu$. But it is easily checked that $A_E(m)$ is not relatively norm compact for any E of strictly positive measure, so that m is not the indefinite integral of any Bochner integrable function.

Our next example shows that hypotheses (1) and (3) (or (3')) do not imply hypothesis (2).

EXAMPLE 6.2 [1, Example 7, p. 377]. Let B be an infinite dimensional Hilbert space, and let $\{e_n\}_{n=1}^\infty$ be an orthonormal system in B . Define a B -valued measure, m , by

$$m(E) = \sum_{n=1}^{\infty} (2^n/n)\mu(E \cap (2^{-n}, 2^{-n+1}])e_n.$$

Then m is clearly μ -continuous, and $A_{(t,1]}(m)$ is a bounded subset of a finite dimensional subspace of B for any $t>0$, so that m satisfies hypotheses (3) and (3'). But $|m|([0, 1])=\sum_{n=1}^\infty 1/n=\infty$, so that hypothesis (2) is not satisfied. We remark that if χ_n denotes the characteristic function of the interval $(2^{-n}, 2^{-n+1}]$, then m is the indefinite integral of the function

$$f = \sum_{n=1}^{\infty} (2^n/n)\chi_n e_n,$$

which is Birkhoff integrable [1] but not Bochner integrable, since $\|f(\cdot)\|$ is not Lebesgue integrable.

Our next example shows that hypotheses (2) and (3) (or (3')) do not imply hypothesis (1).

EXAMPLE 6.3. Let B be the real numbers, and let m be the unit mass at the point 0. Then hypothesis (2) is clearly satisfied, and $A_{(t,1]}(m)=\{0\}$ for any $t>0$ so that hypotheses (3) and (3') are satisfied, but hypothesis (1) is clearly not satisfied.

Our next example serves two purposes. First, it shows that to characterize the indefinite integrals of even bounded Bochner integrable functions the word "somewhere" in hypotheses (3) and (3') can not be omitted. Secondly it shows that the hypotheses of Phillips' Radon-Nikodym theorem [14, p. 134, Theorem 5.5], while sufficient, are not necessary for a measure to be the indefinite integral of a Bochner integrable function. Specifically, Phillips' hypothesis that for any constant N the set

$$K_N = \{m(E)/\mu(E) : E \in S \text{ and } \|m(E)/\mu(E)\| \leq N\}$$

should be weakly relatively compact, is not satisfied by the following example for $N \geq 1$.

EXAMPLE 6.4. Let B be any nonreflexive Banach space, e.g. l^1 , so that the unit ball of B is not weakly compact. By Eberlein's theorem [8, p. 430] the unit ball of B is not weakly sequentially compact, so there exists a sequence $\{b_n\}$ of elements of the unit ball of B which is not weakly relatively compact. For example, in l^1 let b_n be the unit mass at the integer n . Define the function f on X by

$$f = \sum_{n=1}^{\infty} \chi_n b_n$$

where χ_n has the same meaning as in Example 6.2. Then f is a bounded Bochner integrable function. But if $m = \mu_f$, then $A_X(m)$, which is also the same as the set K_N for $N \geq 1$, contains $\{b_n\}$, and so is not weakly relatively compact, much less norm relatively compact.

In our next example we construct a measure which satisfies hypotheses (1) and (2) of the Main Theorem and whose range is relatively compact, but which is not the indefinite integral of a Bochner integrable function on any subset of X of strictly positive measure. In particular, this shows that in hypothesis (3') some condition must be imposed on K , such as disjointness from 0.

EXAMPLE 6.5. Let $B = l^\infty$. Let $\{f_i\}$ be a uniformly bounded sequence of real-valued functions on X which converges to 0 in $L^1(X, \mu)$ and which has the property that if E is any subset of X then the restrictions of the f_i to E are not equicontinuous at any nonisolated point of E . For example let the f_i be the characteristic functions of the intervals $[0, 1/2]$, $[1/2, 1]$, $[0, 1/3]$, $[1/3, 2/3]$, etc. [10, p. 94, Exercise 6]. Define an l^∞ -valued measure, m , on X by

$$(m(E))_i = \int_E f_i d\mu.$$

Then it is clear that m is μ -continuous. Also $|m| \leq c\mu$ where c is a uniform bound for the f_i , so $|m|$ is finite. Furthermore the range of m is contained in the set

$$\{a \in l^\infty : |a_i| \leq \|f_i\|_1\}$$

which is totally bounded and so compact, since $\|f_i\|_1$ converges to 0.

Finally, we indicate why m can not be the indefinite integral of any Bochner integrable function on any set E of strictly positive measure. If m were the indefinite integral of some Bochner integrable function, g , on E then it is not difficult to verify that g must equal f a.e. on E , where f is defined by $(f(x))_i = f_i(x)$. But f is not measurable on E , for if it were, then by Lusin's theorem [10] there would be a compact subset, C , of E of measure arbitrarily close to $\mu(E)$ such that the restriction of f to C is continuous, that is, the f_i are equicontinuous on C . Since C must have nonisolated points, this contradicts the way in which the f_i were chosen.

It is natural to ask whether the analogue of hypothesis (3a) for direction is a necessary condition, that is, whether the indefinite integral of a Bochner integrable function locally almost has compact direction. Our next example shows that this need not be the case.

EXAMPLE 6.6. Let H and $\{e_n\}$ be as in Example 6.2. For each n let g_n be the function on X defined by

$$g_n(x) = (-1)^k \quad \text{for } k/2^n \leq x < (k+1)/2^n$$

where $0 \leq k < 2^n$. Let

$$f = \sum_{n=1}^{\infty} (1/8^n) g_n e_n.$$

Then it is not difficult to show that for any $E \subseteq X$ with $\mu(E) > 1/2$, the range of μ_f restricted to E does not have compact direction.

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